# MATH 20D Spring 2023 Lecture 7.

## Linear Independence

# Outline

# Announcements

- Join the class Zulip via the link available in Canvas

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
    - Make sure you have a scientific calculator ready for next Wednesday.

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
  - Make sure you have a scientific calculator ready for next Wednesday.
  - Assessable content:

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
  - Make sure you have a scientific calculator ready for next Wednesday.
  - Assessable content:
    (a) Material from homeworks 1,2 & 3 (optional questions not included).

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
  - Make sure you have a scientific calculator ready for next Wednesday.
  - Assessable content:
    (a) Material from homeworks 1,2 & 3 (optional questions not included).
    (b) Examples from lectures $1 - 9$ (optional slides not included).

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
  - Make sure you have a scientific calculator ready for next Wednesday.
  - Assessable content:
    (a) Material from homeworks 1,2 & 3 (optional questions not included).
    (b) Examples from lectures $1-9$ (optional slides not included).
    (c) Problem types appearing on review sheet (coming soon with solutions).

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
  - Make sure you have a scientific calculator ready for next Wednesday.
  - Assessable content:
    (a) Material from homeworks 1,2 & 3 (optional questions not included).
    (b) Examples from lectures $1-9$ (optional slides not included).
    (c) Problem types appearing on review sheet (coming soon with solutions).
  - You will be provided with a a table of standard integrals and trigonometric formulas.

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
  - Make sure you have a scientific calculator ready for next Wednesday.
  - Assessable content:
    (a) Material from homeworks 1,2 & 3 (optional questions not included).
    (b) Examples from lectures $1 - 9$ (optional slides not included).
    (c) Problem types appearing on review sheet (coming soon with solutions).
  - You will be provided with a a table of standard integrals and trigonometric formulas.
  - You may bring one double sided sheet of handwritten notes to the exam.

# Announcements

- Join the class Zulip via the link available in Canvas
- Midterm $1$ is approaching (next Wednesday WLH 2005 during lecture)
  - Make sure you have a scientific calculator ready for next Wednesday.
  - Assessable content:
    (a) Material from homeworks 1,2 & 3 (optional questions not included).
    (b) Examples from lectures $1 - 9$ (optional slides not included).
    (c) Problem types appearing on review sheet (coming soon with solutions).
  - You will be provided with a a table of standard integrals and trigonometric formulas.
  - You may bring one double sided sheet of handwritten notes to the exam.
  - Warning! Our lecture stream is not coordinated with the other MATH 20D lecture streams. Question types on the C00 MATH 20D Midterm may differ significantly from those asked in other 20D lecture streams.

# Contents

## Example

- Initially a tank contains 180 litres of solution which is **10%** nitric acid
- At time $t = 0$ a nitric acid solution begins to flow into the tank at a constant rate of 6L/min.
- The solution entering the tank is 20% nitric acid.
- The solution inside the tank is kept well stirred and flows out of the tank at a rate of 6L/min.

(a) Determine the volume of nitric acid in the tank after $10$ minutes.
*Express your answer to the nearest 0.01L*

**Example**

- Initially a tank contains 180 litres of solution which is **10%** nitric acid
- At time $t = 0$ a nitric acid solution begins to flow into the tank at a constant rate of 6L/min.
- The solution entering the tank is 20% nitric acid.
- The solution inside the tank is kept well stirred and flows out of the tank at a rate of 6L/min.

(a) Determine the volume of nitric acid in the tank after $10$ minutes. *Express your answer to the nearest 0.01L*

- After 10 minutes a gushing leak develops and the rate of outflow from the tank increases to 12L/min

## Example

- Initially a tank contains 180 litres of solution which is **10%** nitric acid
- At time $t = 0$ a nitric acid solution begins to flow into the tank at a constant rate of 6L/min.
- The solution entering the tank is 20% nitric acid.
- The solution inside the tank is kept well stirred and flows out of the tank at a rate of 6L/min.

(a) Determine the volume of nitric acid in the tank after $10$ minutes. *Express your answer to the nearest 0.01L*

- After 10 minutes a gushing leak develops and the rate of outflow from the tank increases to 12L/min

(b) Determine the volume of nitric acid in the tank after $10$ minutes after the leak develops. *Express your answer to the nearest 0.01L*

## Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

## Solution I

### Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

Substituting $N(0) = 180 \cdot \frac{1}{10} = 18$ we find that $A = 18$. Hence $N(10) = 23.1024$.

## Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

Substituting $N(0) = 180 \cdot \frac{1}{10} = 18$ we find that $A = 18$. Hence $N(10) = 23.1024$.

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

## Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

Substituting $N(0) = 180 \cdot \frac{1}{10} = 18$ we find that $A = 18$. Hence $N(10) = 23.1024$.

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = \exp\left(\int \frac{2dt}{40 - t}\right) =$$

## Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

Substituting $N(0) = 180 \cdot \frac{1}{10} = 18$ we find that $A = 18$. Hence $N(10) = 23.1024$.

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40-t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = \exp\left(\int \frac{2dt}{40-t}\right) = \exp(-2\log|40-t|) =$$

## Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

Substituting $N(0) = 180 \cdot \frac{1}{10} = 18$ we find that $A = 18$. Hence $N(10) = 23.1024$.

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = \exp\left(\int \frac{2dt}{40 - t}\right) = \exp(-2 \log|40 - t|) = \exp(\log|40 - t|^{-2})$$

## Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

Substituting $N(0) = 180 \cdot \frac{1}{10} = 18$ we find that $A = 18$. Hence $N(10) = 23.1024$.

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = \exp\left(\int \frac{2dt}{40 - t}\right) = \exp(-2\log|40 - t|) = \exp(\log|40 - t|^{-2})$$

So $\mu(t) = (40 - t)^{-2}$ for all $t \leqslant 40$.

## Part (a)

For $t \leqslant 10$, separation of variables applied to $\frac{dN}{dt} = \frac{dN_{\text{in}}}{dt} - \frac{dN_{\text{out}}}{dt}$ gave

$$N = 36 - Ae^{-t/30}.$$

Substituting $N(0) = 180 \cdot \frac{1}{10} = 18$ we find that $A = 18$. Hence $N(10) = 23.1024$.

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = \exp\left(\int \frac{2dt}{40 - t}\right) = \exp(-2\log|40 - t|) = \exp(\log|40 - t|^{-2})$$

So $\mu(t) = (40 - t)^{-2}$ for all $t \leqslant 40$. So $\frac{d}{dt}((40 - t)^{-2}N) = 1.2 \cdot (40 - t)^{-2}$.

## Solution II

### Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = (40 - t)^{-2}$$

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40-t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = (40-t)^{-2}$$

## Part (b) (continued)

So $\frac{d}{dt}((40-t)^{-2}N) = 1.2 \cdot (40-t)^{-2}$ and

$$(40-t)^{-2}N$$

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = (40 - t)^{-2}$$

## Part (b) (continued)

So $\frac{d}{dt}((40 - t)^{-2}N) = 1.2 \cdot (40 - t)^{-2}$ and

$$(40 - t)^{-2}N = 1.2 \cdot \int (40 - t)^{-2} dt$$

## Solution II

### Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40-t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = (40-t)^{-2}$$

### Part (b) (continued)

So $\frac{d}{dt}((40-t)^{-2}N) = 1.2 \cdot (40-t)^{-2}$ and

$$(40-t)^{-2}N = 1.2 \cdot \int (40-t)^{-2} dt = 1.2 \cdot (40-t)^{-1} + C$$

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40-t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = (40-t)^{-2}$$

## Part (b) (continued)

So $\frac{d}{dt}((40-t)^{-2}N) = 1.2 \cdot (40-t)^{-2}$ and

$$(40-t)^{-2}N = 1.2 \cdot \int (40-t)^{-2} dt = 1.2 \cdot (40-t)^{-1} + C$$

So

$$N(t) = 1.2 \cdot (40-t) + C(40-t)^2.$$

## Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40 - t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = (40 - t)^{-2}$$

## Part (b) (continued)

So $\frac{d}{dt}((40-t)^{-2}N) = 1.2 \cdot (40-t)^{-2}$ and

$$(40-t)^{-2}N = 1.2 \cdot \int (40-t)^{-2} dt = 1.2 \cdot (40-t)^{-1} + C$$

So

$$N(t) = 1.2 \cdot (40-t) + C(40-t)^2.$$

The initial condition $N(10) = 23.1024$ gives $C = -0.0143$.

### Part (b)

For $t \geqslant 10$, the volume of nitric acid varies according to

$$\frac{dN}{dt} + \frac{2N}{40-t} = 1.2$$

The integrating factor is then given by

$$\mu(t) = (40-t)^{-2}$$

### Part (b) (continued)

So $\frac{d}{dt}((40-t)^{-2}N) = 1.2 \cdot (40-t)^{-2}$ and

$$(40-t)^{-2}N = 1.2 \cdot \int (40-t)^{-2} dt = 1.2 \cdot (40-t)^{-1} + C$$

So

$$N(t) = 1.2 \cdot (40-t) + C(40-t)^2.$$

The initial condition $N(10) = 23.1024$ gives $C = -0.0143$. So $N(20) = 18.27$L.

# Contents

### Definition

*A second order linear ODE can be written in the form*

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = g(t) \tag{1}$$

*where* $a(t), b(t), c(t)$, *and* $g(t)$ *are functions that only depend on* $t$.

**Definition**

*A second order linear ODE can be written in the form*

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = g(t) \tag{1}$$

*where $a(t), b(t), c(t)$, and $g(t)$ are functions that only depend on $t$.*

- For simplicity we begin by studying ODE's of the form

$$ay''(t) + by'(t) + cy(t) = 0 \tag{2}$$

where $a$, $b$, and $c$ are all constant and $a \neq 0$.

**Definition**

*A second order linear ODE can be written in the form*

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = g(t) \tag{1}$$

*where $a(t), b(t), c(t)$, and $g(t)$ are functions that only depend on $t$.*

- For simplicity we begin by studying ODE's of the form

$$ay''(t) + by'(t) + cy(t) = 0 \tag{2}$$

  where $a$, $b$, and $c$ are all constant and $a \neq 0$.

- The ODE governing the **displacement** $y(t)$ for the harmonic oscillator is

$$my''(t) + by'(t) + ky(t) = 0$$

  where $m$ is the **mass** of the object attached to the spring, $b \geqslant 0$ is coefficient of **friction**, and $k > 0$ measures the **stiffness** of the spring.

- Before considering the 2nd order constant coefficient ODE

$$ay''(t) + by'(t) + cy(t) = 0$$

is helpful to study it's first order counter part

$$y'(t) + ky(t) = 0 \tag{3}$$

which has a general solution $y(t) = Ae^{-kt}$ where $A \in \mathbb{R}$ is constant.

- Before considering the 2nd order constant coefficient ODE

$$ay''(t) + by'(t) + cy(t) = 0$$

is helpful to study it's first order counter part

$$y'(t) + ky(t) = 0 \tag{3}$$

which has a general solution $y(t) = Ae^{-kt}$ where $A \in \mathbb{R}$ is constant.

**Definition**

*Let $I \subset \mathbb{R}$ be a domain. We say that function $y_1, y_2 : I \to \mathbb{R}$ are **linearly dependent** if there exists a constant $\alpha \in \mathbb{R}$ such that*

$$y_1(t) = \alpha y_2(t)$$

*for all $t \in I$. If not, we say that $y_1$ and $y_2$ are **linearly independent***

- Before considering the 2nd order constant coefficient ODE

$$ay''(t) + by'(t) + cy(t) = 0$$

is helpful to study it's first order counter part

$$y'(t) + ky(t) = 0 \tag{3}$$

which has a general solution $y(t) = Ae^{-kt}$ where $A \in \mathbb{R}$ is constant.

**Definition**

*Let $I \subset \mathbb{R}$ be a domain. We say that function $y_1, y_2 : I \to \mathbb{R}$ are* ***linearly dependent*** *if there exists a constant $\alpha \in \mathbb{R}$ such that*

$$y_1(t) = \alpha y_2(t)$$

*for all $t \in I$. If not, we say that $y_1$ and $y_2$ are* ***linearly independent***

**Example**

Show that the ODE $y'' + 5y' - 6y = 0$ admits two linearly independent solutions.

### Theorem

*Let* $a \neq 0$, $b$, *and* $c$ *be constants and consider the ODE*

$$ay''(t) + by'(t) + cy(t) = 0 \tag{4}$$

### Theorem

*Let $a \neq 0$, $b$, and $c$ be constants and consider the ODE*

$$ay''(t) + by'(t) + cy(t) = 0 \tag{4}$$

- *Then (4) admits a pair of linearly independent solutions.*

### Theorem

*Let $a \neq 0$, $b$, and $c$ be constants and consider the ODE*

$$ay''(t) + by'(t) + cy(t) = 0 \tag{4}$$

- *Then (4) admits a pair of linearly independent solutions.*
- *Suppose $y_1$ and $y_2$ are linearly independent solutions to (4). Then*

$$y\colon \mathbb{R} \to \mathbb{R}, \quad y(t) = C_1 y_1(t) + C_2 y_2(t) \tag{5}$$

  *is a **general solution** to (4).*

**Theorem**

*Let $a \neq 0$, $b$, and $c$ be constants and consider the ODE*

$$ay''(t) + by'(t) + cy(t) = 0 \tag{4}$$

- *Then (4) admits a pair of linearly independent solutions.*
- *Suppose $y_1$ and $y_2$ are linearly independent solutions to (4). Then*

$$y\colon \mathbb{R} \to \mathbb{R}, \quad y(t) = C_1 y_1(t) + C_2 y_2(t) \tag{5}$$

*is a **general solution** to (4). More precisely, if $x_0, y_0, y_1 \in \mathbb{R}$, then there exists unique constants $C_1$ and $C_2$ such that (5) defines a solution to (4) satisfying*

$$y(x_0) = y_0 \qquad \text{and} \qquad y'(x_0) = y_1.$$

**Theorem**

*Let $a \neq 0$, $b$, and $c$ be constants and consider the ODE*

$$ay''(t) + by'(t) + cy(t) = 0 \tag{4}$$

- *Then (4) admits a pair of linearly independent solutions.*
- *Suppose $y_1$ and $y_2$ are linearly independent solutions to (4). Then*

$$y\colon \mathbb{R} \to \mathbb{R}, \quad y(t) = C_1 y_1(t) + C_2 y_2(t) \tag{5}$$

*is a **general solution** to (4). More precisely, if $x_0, y_0, y_1 \in \mathbb{R}$, then there exists unique constants $C_1$ and $C_2$ such that (5) defines a solution to (4) satisfying*

$$y(x_0) = y_0 \qquad \text{and} \qquad y'(x_0) = y_1.$$

**Example**

We've seen that $y_1(t) = e^t$ and $y_2(t) = e^{-6t}$ are linearly independent solutions to $y'' + 5y' - 6y = 0$. The theorem implies $y(t) = C_1 e^t + C_2 e^{-6t}$ is a general solution.

### Definition

*Given constants $a \neq 0$, $b$, and $c$, we call the equation*

$$ar^2 + br + c = 0 \tag{6}$$

*the **auxilliary or characteristic equation** of the ODE $ay'' + by' + cy = 0$.*

**Definition**

*Given constants $a \neq 0$, $b$, and $c$, we call the equation*

$$ar^2 + br + c = 0 \tag{6}$$

*the **auxilliary or characteristic equation** of the ODE $ay'' + by' + cy = 0$.*

- If $r$ is a root of (6) then $y(t) = e^{rt}$ solves the ODE $ay'' + by' + cy = 0$.

**Definition**

*Given constants $a \neq 0$, $b$, and $c$, we call the equation*

$$ar^2 + br + c = 0 \tag{6}$$

*the* ***auxilliary or characteristic equation*** *of the ODE $ay'' + by' + cy = 0$.*

- If $r$ is a root of (6) then $y(t) = e^{rt}$ solves the ODE $ay'' + by' + cy = 0$.
- If $b^2 - 4ac > 0$ so that

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are **distinct reals roots** of (6),

**Definition**

*Given constants $a \neq 0$, $b$, and $c$, we call the equation*

$$ar^2 + br + c = 0 \tag{6}$$

*the **auxilliary or characteristic equation** of the ODE $ay'' + by' + cy = 0$.*

- If $r$ is a root of (6) then $y(t) = e^{rt}$ solves the ODE $ay'' + by' + cy = 0$.
- If $b^2 - 4ac > 0$ so that

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are **distinct reals roots** of (6),then

$$y_1(t) = e^{r_1 t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t}$$

are **linearly independent** solutions to $ay'' + by' + cy = 0$.

**Definition**

*Given constants $a \neq 0$, $b$, and $c$, we call the equation*

$$ar^2 + br + c = 0 \tag{6}$$

*the **auxilliary or characteristic equation** of the ODE $ay'' + by' + cy = 0$.*

- If $r$ is a root of (6) then $y(t) = e^{rt}$ solves the ODE $ay'' + by' + cy = 0$.
- If $b^2 - 4ac > 0$ so that

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

  are **distinct reals roots** of (6),then

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t}$$

  are **linearly independent** solutions to $ay'' + by' + cy = 0$.
- So if $b^2 - 4ac > 0$ then the theorem on the previous slide implies that $ay'' + by' + cy = 0$ has the **general solution** $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ with $t \in \mathbb{R}$.

How can we construct **two linearly** independent solutions to the ODE

$$ay''(t) + by'(t) + c = 0$$

when the auxilliary equation $ar^2 + br + c = 0$ has a repeated real root?

***Answer:*** *Trial a solution of the form* $y(t) = v(t) \cdot e^{rt}$*.*

How can we construct **two linearly** independent solutions to the ODE

$$ay''(t) + by'(t) + c = 0$$

when the auxilliary equation $ar^2 + br + c = 0$ has a repeated real root?

***Answer:*** *Trial a solution of the form* $y(t) = v(t) \cdot e^{rt}$.

### Theorem

*Suppose* $b^2 - 4ac = 0$ *so that* $ar^2 + br + c = 0$ *has a unique real root* $r = \frac{-b}{2a}$.

How can we construct **two linearly** independent solutions to the ODE

$$ay''(t) + by'(t) + c = 0$$

when the auxilliary equation $ar^2 + br + c = 0$ has a repeated real root?

***Answer:*** *Trial a solution of the form $y(t) = v(t) \cdot e^{rt}$.*

**Theorem**

*Suppose $b^2 - 4ac = 0$ so that $ar^2 + br + c = 0$ has a unique real root $r = \frac{-b}{2a}$. Then*

$$y_1(t) = e^{rt} \qquad \text{and} \qquad y_2(t) = te^{rt}$$

*are linearly independent solution to $ay'' + by' + cy = 0$.*

How can we construct **two linearly** independent solutions to the ODE

$$ay''(t) + by'(t) + c = 0$$

when the auxilliary equation $ar^2 + br + c = 0$ has a repeated real root?

***Answer:*** *Trial a solution of the form* $y(t) = v(t) \cdot e^{rt}$.

### Theorem

*Suppose* $b^2 - 4ac = 0$ *so that* $ar^2 + br + c = 0$ *has a unique real root* $r = \frac{-b}{2a}$. *Then*

$$y_1(t) = e^{rt} \qquad \text{and} \qquad y_2(t) = te^{rt}$$

*are linearly independent solution to* $ay'' + by' + cy = 0$.

### Example

(a) Write down a general solution to the ODE $y'' - 4y' + 4y = 0$.

(b) Solve the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$